Dealing with Resistance to Flow into Surface Waters

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The Issue

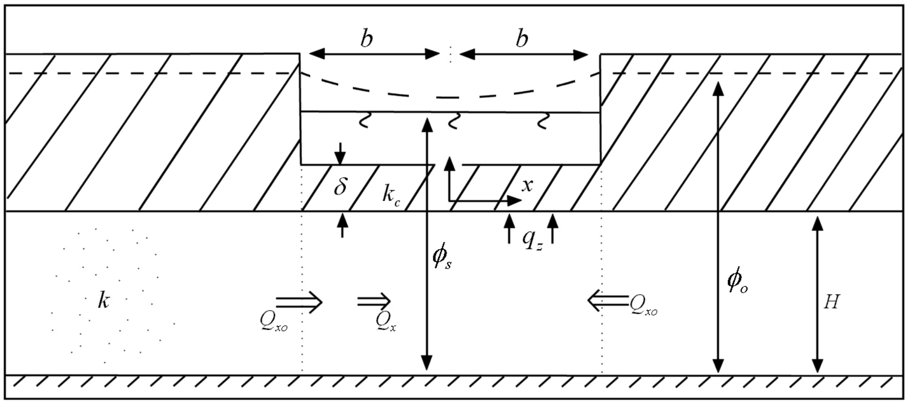
Streams or lakes often have a layer of silt and organic material on the bottom, creating resistance to flow from the aquifer into the stream or lake, or vice versa. In addition, the stream or lake may not be in direct contact with the aquifer, but the stream or lake bottom may be separated from the aquifer top by a low permeable layer (e.g. clay). The resistance to flow through such a low permeable stream bottom or lake bottom may locally be of importance to the groundwater flow regime. For instance, a well that withdraws water from an aquifer near a stream or lake may receive some of its water from that surface water body. The precise amount depends, among other things, upon the resistance of the bottom layer of the stream or lake. To properly represent this effect in GFLOW, the line-sinks used to model the stream may be given a resistance parameter and a width. In W*h*AEM , however, the line-sinks do not have these attributes, but it is still possible to take the effect of bottom resistance into account. The analysis below leads to some rules of thumb regarding the use of line-sinks with or without a resistance parameter in representing streams or lakes with bottom resistance.

Mathematical Description of Flow

In Figure a schematic cross section over a stream and the underlying aquifer is depicted. The stream has a resistance layer which separates the stream channel from the aquifer. The resistance layer has a thickness δ $[m]$ and hydraulic conductivity k_c $[m/day]$, which translates in a resistance to flow c $[days]$ as

$$c = \frac{\delta}{k_c} \tag{1}$$

The width of the stream is $2b$, the water level in the stream is ϕ_s, and the head in the aquifer at the stream boundary is ϕ_0. The head underneath the stream ϕ varies across the stream width: $\phi = \phi(x)$. For the case of one-dimensional flow



captionConceptual model of a stream with a bottom resistance layer.

the head ϕ underneath an aquitard is given by Verruijt, 1970 (page 30, equation 4.8) as:

$$\phi = \phi_s + Ae^{x/\lambda} + Be^{-x/\lambda} \tag{2}$$

where A and B are obtained from boundary conditions. The parameter λ $[m]$ is the "leakage factor" or "characteristic leakage length" defined as:

$$\lambda = \sqrt{kHc} \tag{3}$$

where k is the hydraulic conductivity in the aquifer and H is the aquifer thickness. The total flow integrated over the aquifer height underneath the stream bottom follows from Darcy's law:

$$Q_x = -kH\frac{d\phi}{dx} \tag{4}$$

which becomes with (2):

$$Q_x = -\frac{kH}{\lambda}\left(Ae^{x/\lambda} - Be^{-x/\lambda}\right) \tag{5}$$

The boundary conditions necessary to resolve A and B are:

$$x = 0 \quad ; \quad Q_x = 0 \qquad \text{and} \qquad x = b \quad ; \quad \phi = \phi_0 \tag{6}$$

Substituting the condition at $x = 0$ in (5) yields that $A = B$. Substituting the condition at $x = b$ in (2) leads to:

$$A = B = \frac{\Delta\phi}{2\cosh(b/\lambda)} \tag{7}$$

where $\Delta\phi$ is defined as the difference between the water level in the stream and the head in the aquifer at $x = b$ or $x = -b$:

$$\Delta\phi = \phi_0 - \phi_s \tag{8}$$

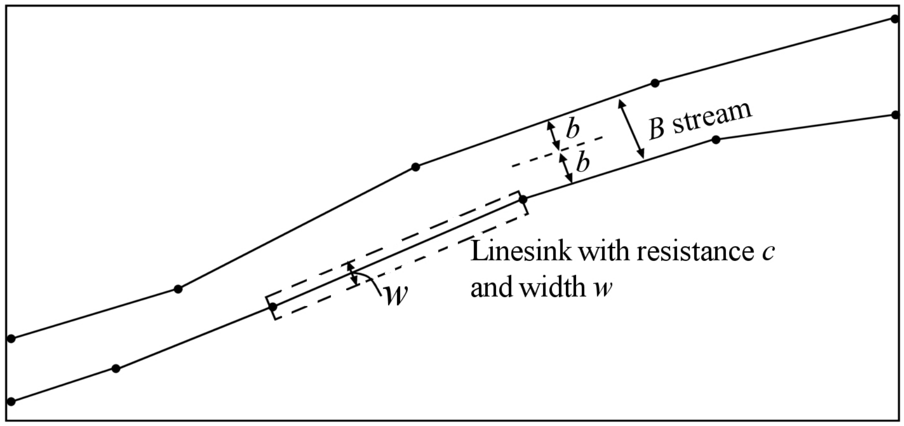


Figure 1: A stream modeled by two line-sink strings on either stream boundary.

Combining (7) with (5) yields the following expression for the flow Q_x in the aquifer underneath the stream:

$$Q_x = -\frac{kH\Delta\phi}{\lambda}\frac{\sinh(x/\lambda)}{\cosh(b/\lambda)} \tag{9}$$

Combining (7) with (2) yields the follow expression for the head ϕ in the aquifer underneath the stream.

$$\phi(x) = \phi_s + \Delta\phi\frac{\cosh(x/\lambda)}{\cosh(b/\lambda)} \tag{10}$$

Representing the Stream with Line-Sinks

The following analysis has been adapted from a presentation by O. D. L. Strack of the University of Minnesota given at the *Third International Conference on the Analytic Element Method in Modeling Groundwater Flow, Brainerd, MN, April 19 - 21, 2000.*

In Figure 1 a stream is shown in plan view with line-sink strings arranged along each of the two stream boundaries. The line-sinks are given a width w and a resistance c to represent the resistance to flow from the aquifer into the stream that results from the bottom layer, see Figure . The resistance c can be computed from (1). The question is what value to assign to the line-sink width w. In Figure 2 a cross section is provided over the aquifer and the two line-sinks with width w. The total inflow in each line-sink, per unit length perpendicular to the plane of Figure 2 follows from Darcy's law in the vertical direction:

$$q_z w = \frac{\Delta\phi}{c}w = \frac{kH\Delta\phi}{\lambda^2}w \tag{11}$$

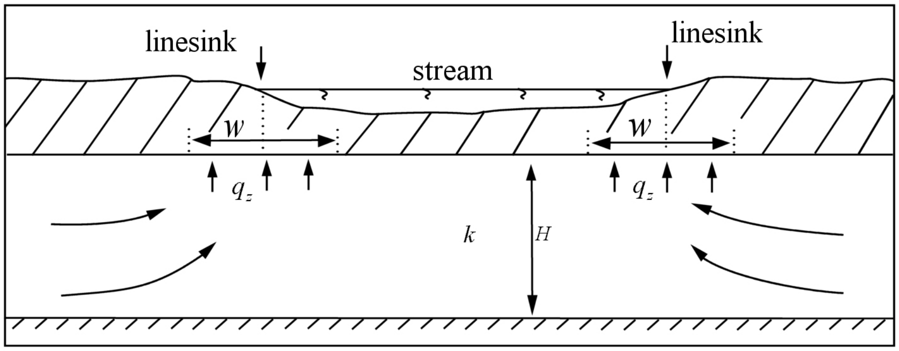


Figure 2: Cross section over the aquifer and the line-sinks representing the stream.

This inflow rate must be equal to the flow Q_{x0} in the aquifer at $x = b$ or $x = -b$ in Figure , which follows from (9) by setting $x = b$ or $x = -b$. This leads to the following expression for Q_{x0} at $x = -b$:

$$Q_{x0} = \frac{kH\Delta\phi}{\lambda}\tanh(b/\lambda) \tag{12}$$

Setting (11) equal to (12) yields for w:

$$w = \lambda \tanh(b/\lambda) \tag{13}$$

For values of λ much smaller than b expression (13) becomes:

$$w = \lambda \qquad\qquad \lambda \ll b \tag{14}$$

Line-sinks without resistance

In the event the line-sink cannot be given a resistance, as is the case for the groundwater flow model WhAEM , the same resistance to flow can be simulated by moving the line-sink string inward from the stream boundary over a distance λ, provided that $\lambda \ll b$. This may be seen as follows. Consider the flow Q_{x0} from the stream boundary to the line-sink inside of that boundary:

$$Q_{x0} = -kH\frac{\Delta\phi}{d} \tag{15}$$

where d is the distance between the stream boundary and the line-sink. The head difference $\Delta\phi$ is the same as in (8) with ϕ_s the head at the line-sink and ϕ_0 the head at the stream boundary. The two flows in (12) and (15) must be the same, which results in the following expression for d:

$$d = \frac{\lambda}{\tanh(b/\lambda)} \tag{16}$$

In case λ is much smaller than b this results in:

$$d = \lambda \qquad\qquad \lambda \ll b \tag{17}$$

The parameter b is the half width of the stream, see Figure . Defining the total width of the stream $B = 2b$ the condition $\lambda \ll b$ may, for practical purposes, be expressed in terms of the width B as:

$$\lambda < 0.1B \tag{18}$$

Rules of Thumb

Some rules of thumb are presented to facilitate the representation of streams or lakes by line-sink strings in a manner that ensures the proper inflow rates into the stream or lake. The rules of thumb prescribe the location of the line-sinks and, if supported by the model, the choice of the resistance and width parameters for these line-sinks.

Using line-sinks with resistance

In Figure 1 a stream is modeled with two line-sink strings. The line-sinks are positioned at the stream boundary. The line-sinks have a resistance parameter c and a width parameter w.

The first step is to calculate the resistance of the bottom layer of the stream using (1):

$$c = \frac{\delta}{k_c} \tag{19}$$

Next the characteristic leakage length λ must be calculated using (3)

$$\lambda = \sqrt{kHc} \tag{20}$$

In the following rules the actual stream width is B, while the width parameter for the line-sinks located along the stream boundaries is w. In the event that λ is smaller than $0.1B$ the width w for a line-sink on the boundary of the stream follows from:

$$w = \lambda \qquad\qquad \lambda \leq 0.1B \tag{21}$$

In the event that λ is larger than $0.1B$ the width w for the line-sinks on the boundary of the stream follows from:

$$w = \lambda \tanh(B/2\lambda) \qquad\qquad 0.1B < \lambda < 2B \tag{22}$$

In the event that λ is larger than $2B$ the width w for the line-sinks on the boundary of the stream follows from:

$$w = B/2 \qquad\qquad \lambda \geq 2B \tag{23}$$

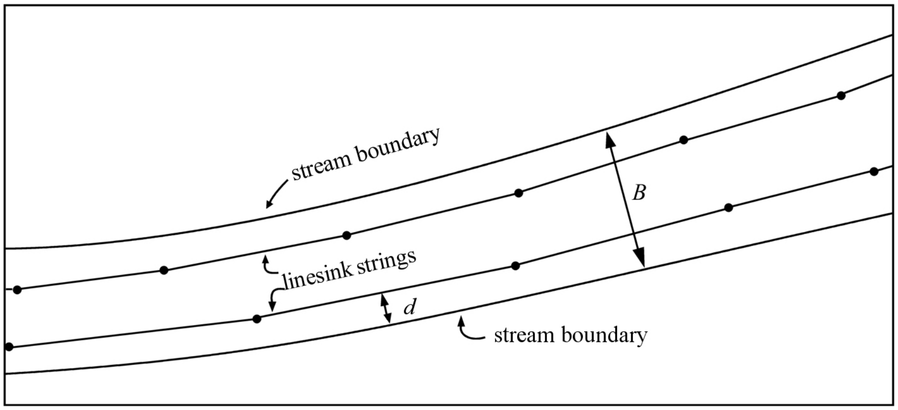


Figure 3: Modeling a stream using line-sinks without resistance.

If the stream is not in the immediate proximity of other boundary conditions in the aquifer, for instance one or more wells, the exact location of the stream boundary is less critical and the stream may be represented by a single line-sink string at its axis with width w equal to double the width calculated from expressions (21) through (23). For the case of a large characteristic leakage length this implies setting the line-sink width equal to the actual stream width:

$$w = B \qquad \lambda > 2B \tag{24}$$

Using line-sinks without resistance

In Figure 3 a stream is modeled with two line-sink strings near each stream boundary. The line-sinks are head specified without resistance or width parameters, as found in WhAEM . To simulate the proper resistance to flow the position of the line-sink string, relative to the stream boundaries, may be manipulated. As seen in Figure 3 the line-sinks are placed at a distance d from the stream boundary. This distance may be calculated as follows:

$$d = \lambda \qquad \lambda \leq 0.1B \tag{25}$$

$$d = \frac{\lambda}{\tanh(B/2\lambda)} \qquad \lambda > 0.1B \tag{26}$$

Shifting line-sinks inward, away from the stream boundaries, over a distance d as calculated by (26) may lead to unrealistic situations. For instance, if d is larger than $B/2$ the line-sinks would be moved past the stream axis. However, if the purpose of this relocation exercise is to force the proper interaction between the well and the stream *in terms of the groundwater flow rates* it may be sufficient to move only the line-sink string on the side of the stream opposite the well. In this manner, the well will draw the proper (at least a more realistic) amount of water from the stream. While the flow patterns between the well and the

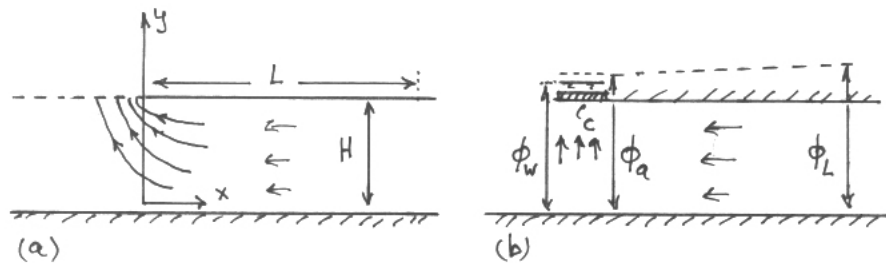


Figure 4: Resistance to flow in the vertical plane (a) is replaced by resistance to flow through a fictitious resistance layer(b).

stream are not realistic, the amount of flow is. In particular, this will improve the size and shape of the capture zone upgradient from the well and the stream. On the other hand if the distance d over which the line-sinks have to be shifted becomes too large, for instance across the opposite stream boundary ($d > B$, which occurs for $\lambda > 0.65B$), this "trick" to simulate bottom resistance becomes problematic. For the case of capture zone delineation it may be possible to move the well away from the stream instead of the line-sinks away from the well. The distance d over which the well is to be moved follows from (26). However, to obtain reliable solutions for these larger values of λ it is recommended that a groundwater flow model is used that can properly incorporate stream bottom resistance.

Resistance to 3D Flow into the Stream

Previously, only the resistance to flow through stream or lake bottom sediments has been considered. It may be necessary to also consider the resistance to flow due to vertical flow components in the aquifer near and underneath the stream or lake. The issue is illustrated in Figure 4, where flow in a vertical section across a stream and aquifer is depicted for both a three-dimensional conceptual model and a Dupuit-Forchheimer model. The vertical resistance to flow in Figure 4(a) is replaced by the resistance to flow through a fictitious resistance layer in Figure 4(b). The question is how to determine this resistance.

Assuming a wide stream or lake, the solution to the flow problem in Figure 4 is given by Verruijt (1970):

$$e^{\frac{x\pi}{2H}} = \sqrt{\left\{\sinh\left(\frac{k\phi\pi}{2\sigma}\right)\cos\left(\frac{\Psi\pi}{2\sigma}\right)\right\}^2 + \left\{\cosh\left(\frac{k\phi\pi}{2\sigma}\right)\sin\left(\frac{\Psi\pi}{2\sigma}\right)\right\}^2} \quad (27)$$

and

$$\tan\left(\frac{y\pi}{2H}\right) = \frac{\tan\left(\frac{\Psi\pi}{2\sigma}\right)}{\tanh\left(\frac{k\phi\pi}{2\sigma}\right)} \quad (28)$$

where ϕ is the head, Ψ is the stream function, which is constant along streamlines, and σ $[ft^2/day]$ is the total flow over the aquifer height per unit length perpendicular to the plane in Figure 4. We will consider the difference $\Delta\phi$ in head at a point L from the stream or lake boundary and the head at the lake boundary. The head at the stream is found by substituting $x = 0$ and $y = H$ into (27) and (28), respectively, where H is the aquifer thickness, see Figure 4. It follows from (28) with $y = H$ that $\Psi = \sigma$ and it follows from (27) with $x = 0$ and $\Psi = \sigma$ that the head at the stream is zero. It follows from (27) with $x = L$ and $\Psi = \sigma$ that:

$$e^{\frac{L\pi}{2H}} = \cosh\left(\frac{k\Delta\phi}{\sigma}\frac{\pi}{2}\right) \tag{29}$$

which may be rewritten as

$$\sigma = \frac{\Delta\phi}{c^*} \tag{30}$$

where

$$c^* = \frac{2}{\pi k}\text{arccosh}\left(e^{\frac{L}{H}\frac{\pi}{2}}\right) \tag{31}$$

which may also be written as

$$c^* = \frac{2}{\pi k}\ln\left(e^{\frac{L}{H}\frac{\pi}{2}} + \sqrt{e^{\frac{L}{H}\pi} - 1}\right) \tag{32}$$

The flow σ into the line-sink depicted in Figure 4(b) follows from:

$$\sigma = \frac{(\phi_a - \phi_w)w}{c} \tag{33}$$

where ϕ_a and ϕ_w are the heads in the aquifer and line-sink, respectively. The Dupuit-Forchheimer flow toward the line-sinks that define the stream or lake boundary follows from Darcy's law:

$$\sigma = \frac{(\phi_L - \phi_a)kH}{L} \tag{34}$$

where ϕ_L is the head at $x = L$. Eliminating ϕ_a from (33) and (34) yields:

$$\sigma = \frac{\Delta\phi w}{\frac{Lw}{kH} + c} \tag{35}$$

Comparing (35) with (30) and (31) yields for the line-sink resistance:

$$c = \frac{2w}{\pi k}\left\{\ln\left(e^{\frac{L}{H}\frac{\pi}{2}} + \sqrt{e^{\frac{L}{H}\pi} - 1} - \frac{L}{H}\frac{\pi}{2}\right)\right\} \tag{36}$$

or

$$c = \frac{2w}{\pi k}\left\{\ln\left(e^{\frac{L}{H}\frac{\pi}{2}} + \sqrt{e^{\frac{L}{H}\pi} - 1} - \ln\left(e^{\frac{L}{H}\frac{\pi}{2}}\right)\right)\right\} \tag{37}$$

which reduces to

$$c = \frac{2w}{\pi k}\ln\left(1 + \sqrt{1 - e^{-\frac{L}{H}\pi}}\right) \tag{38}$$

If $L >> H$ this becomes

$$c = \frac{2w}{\pi k} \ln 2 \tag{39}$$

or briefly:

$$c = 0.4413 \, \frac{w}{k} \tag{40}$$

The resistance to vertical flow near the stream, as approximated by (40), usually has only a small effect on the heads and flow rates in the aquifer, unless the inflow or outflow from the stream is very large. This may locally be the case, for instance, when pumping a well very close to the stream or lake boundary. In case there is no bottom resistance in the stream or lake the line-sink width w may be set to unity and c follows from (40) with $w = 1$ as $c = 0.4413/k$. In the event that there is some bottom resistance, the line-sink width w must be determined by use of the rules defined earlier in this document (only using the bottom resistance in λ) and the resistance c, calculated from (40), must be added to the bottom resistance before entering it on the *Line-sink Properties* form in GFLOW.

References

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